Scalable and robust regression models for continuous proportional data

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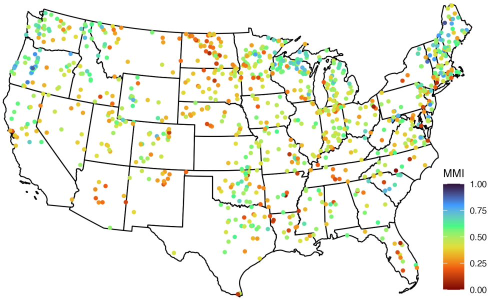
Joint work with B. Dahl (Duke), O. Ovaskainen (U. Jyväskylä), and D. Dunson (Duke)

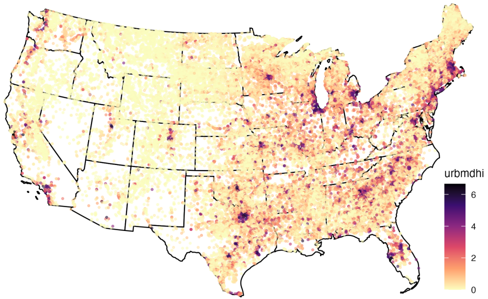
Continuous proportional data

Response Y in the unit interval $[0, 1]$, non-count based proportions

- (Medical imaging) Percentage of tissue area in mammogram [Peplonska et al., 2012]
- (Econometrics) Central-bank independence index [Berggren et al., 2014]
- (Political science) Suffrage (voting rights) index [Kubinec, 2023]
- Especially common in ecology:
  - Percent cover measurements [de Vargas Ribeiro et al., 2022][Korhonen et al., 2024][van Strien et al., 2024]
  - Diversity index [Lindholm et al., 2021][Bharti et al., 2023][Rolls et al., 2023][Qiao et al., 2025]
  - [Warton and Hui, 2011]: $\approx$14% of ecology papers involves non-count based proportions.

Benthic macroinvertebrate multimetric index (MMI)





- Higher MMI indicates a healthier/diverse lake benthic macroinvertebrate community
- (Left): MMI of 949 lakes from 2017 NLA survey [U.S. Environmental Protection Agency, 2022]
- (Right): Lake watershed covariate of 50k+ lakes from LakeCat data [Hill et al., 2018]
  - medium/high urban land cover, soil erodibility factor, coniferous forest cover, ...

Modeling continuous proportional data

- Approach 1: **Transform** $Y \mapsto f(Y) \in \mathbb{R}$ (e.g. logit) and fit normal linear model.
  - Interpretation based on $E(f(Y) \mid X)$, not $E(Y \mid X)$
  - Problematic for boundary values.
- Approach 2: **Beta regression** [Ferrari and Cribari-Neto, 2004, Cribari-Neto and Zeileis, 2010]

$$Y_i \mid \boldsymbol{x}_i \overset{\text{ind}}{\sim} \text{Beta}(\mu_i, \phi), g(\mu_i) = \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta},$$

$$p_{\text{beta}}(y; \mu, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma(\phi - \mu\phi)} y^{\mu\phi - 1}(1-y)^{\phi - \mu\phi - 1}, \quad 0 < y < 1$$

  - Mean & precision parameterization, $E(Y) = \mu$, $\text{var}(Y) = \mu(1-\mu)/(1+\phi)$
  - Familiar GLM-like structure, $g(E(y \mid \boldsymbol{x})) = \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}$, with a choice of link function g.

Limitations of beta regression

- **Non-robustness**: sensitive to violation of beta response assumption.
  - Beta does not belong to a natural exponential family
  - Does not belong to GLM, μ and ϕ not orthogonal to each other [Ferrari and Cribari-Neto, 2004]
- **Computational challenges**: Bayesian inference in hierarchical settings.
  - Mixed models, longitudinal and spatial models: generic methods (e.g. `Stan`) may suffer
  - Existing scalable methods cannot be easily applied, unless relying on approximation
- **Handling boundary values**: cannot handle exact 0s and 1s
  - Preprocess ("nudge") the data to lie between open interval $(0, 1)$ [Smithson and Verkuilen, 2006]
  - Results are often sensitive [Kosmidis and Zeileis, 2024], not desirable

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Summary of contribution

- **Cobin regression**: continuous binomial (cobin) regression model
  - A proper GLM approach based on **exponential dispersion model** [Jørgensen, 1987]
  - Inherits attractive properties of GLM, including robustness of $\hat{\beta}$ to model misspecificaton
- **Micobin regression**: based on dispersion mixtures of cobin distributions
  - More **flexible** & **robust** family of distribution (cf. t dist as scale mixture of normal)
  - **Can handle exact 0s and 1s**, different from structural zeros [Blasco-Moreno et al., 2019]
- We introduce **Kolmogorov-Gamma augmentation** for Bayesian computation
  - Converts cobin/micobin likelihood into **conditionally normal likelihood**
  - Seamless integration with scalable methods for latent Gaussian models

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Exponential dispersion model (1)

- Exponential dispersion model [Jørgensen, 1987] offers a principled way to add dispersion parameter λ^{-1} that controls variance to one-parameter natural exponential family

	exponential tilting by θ $\longrightarrow$	λ-fold convolution and scale by λ^{-1} $\longrightarrow$
$H_1 \sim N(0,1)$ $h_1(y) = (2\pi)^{-1/2}\exp(-y^2/2)$ $y \in \mathbb{R}$	$Y \sim N(\theta, 1)$ $\exp(\theta y - B_1(\theta))h_1(y)$ $\theta \in \mathbb{R}$	$Y \sim N(\theta, \lambda^{-1})$ $\exp(\lambda\theta y - \lambda B_1(\theta))h_1(y,\lambda)$ $\lambda > 0$

$B_1(\theta) = \log E(e^{\theta H_1}) = \frac{\theta^2}{2}$, $\quad h_1(y,\lambda) =$ density of $\frac{1}{\lambda}\sum_{l=1}^{\lambda} H_1^{(l)} = \frac{\exp(-\lambda y^2/2)}{\sqrt{2\pi/\lambda}}$

Exponential dispersion model (2)

- Exponential dispersion model [Jørgensen, 1987] offers a principled way to add dispersion parameter λ^{-1} that controls variance to one-parameter natural exponential family

	exponential tilting by θ $\longrightarrow$	λ-fold convolution and scale by λ^{-1} $\longrightarrow$
$H_2 \sim \text{Exp}(1)$	$Y \sim \text{Exp}(1-\theta)$	$Y \sim \text{Gamma}(\lambda, \lambda(1-\theta))$
$h_2(y) = \exp(-y)$	$\exp(\theta y - B_2(\theta))h_2(y)$	$\exp(\lambda\theta y - \lambda B_2(\theta))h_2(y,\lambda)$
$y \geq 0$	$\theta < 1$	$\lambda > 0$

$B_2(\theta) = \log E(e^{\theta H_2}) = -\log(1-\theta)$, $h_2(y,\lambda) =$ density of $\frac{1}{\lambda}\sum_{l=1}^{\lambda} H_2^{(l)} = \frac{\lambda^\lambda y^{\lambda-1}\exp(-\lambda y)}{\Gamma(\lambda)}$

Exponential dispersion model (3)

- Exponential dispersion model [Jørgensen, 1987] offers a principled way to add dispersion parameter λ^{-1} that controls variance to one-parameter natural exponential family

	exponential tilting by θ $\longrightarrow$	λ-fold convolution and scale by λ^{-1} $\longrightarrow$
$H_3 \sim \mathrm{InvGamma}(1/2, 1/2)$	$Y \sim \mathrm{InvGau}((-2\theta)^{-1/2}, 1)$	$Y \sim \mathrm{InvGau}((-2\theta)^{-1/2}, \lambda)$
$h_3(y) = (2\pi y^3)^{-1/2} \exp(-\frac{1}{2y})$	$\exp(\theta y - B_3(\theta))h_3(y)$	$\exp(\lambda\theta y - \lambda B_3(\theta))h_3(y, \lambda)$
$y > 0$	$\theta < 0$	$\lambda > 0$

$B_3(\theta) = \log E(e^{\theta H_3}) = -(-2\theta)^{1/2}$, $\quad h_3(y, \lambda) =$ density of $\frac{1}{\lambda}\sum_{l=1}^{\lambda} H_3^{(l)} = \frac{\lambda^{1/2}\exp(-\lambda/(2y))}{\sqrt{2\pi y^3}}$

Exponential dispersion model (4)

	exponential tilting by θ $\longrightarrow$	λ-fold convolution and scale by λ^{-1} $\longrightarrow$
$H \sim \mathrm{Unif}(0,1)$	$Y \sim \mathrm{cobin}(\theta, 1)$	$Y \sim \mathrm{cobin}(\theta, \lambda^{-1})$
$h(y) = 1$	$\exp(\theta y - B(\theta))h(y)$	$\exp(\lambda\theta y - \lambda B(\theta))h(y,\lambda)$
$0 \le y \le 1$	$\theta \in \mathbb{R}$	$\lambda \in \mathbb{N}$

$B(\theta) = \log E(e^{\theta H}) = \log\left(\frac{e^\theta - 1}{\theta}\right)$, $h(y,\lambda) =$ density of $\frac{1}{\lambda}\sum_{l=1}^{\lambda} H^{(l)}$ [Bates, 1955]

$$h(y,\lambda) = \frac{\lambda}{(\lambda-1)!}\sum_{k=0}^{\lambda}(-1)^k\binom{\lambda}{k}\{\max(\lambda y - k, 0)\}^{\lambda-1}$$

Continuous binomial distribution

Definition 1. (cobin)

We say $Y \sim \mathrm{cobin}(\theta, \lambda^{-1})$, natural param. $\theta \in \mathbb{R}$, dispersion $\lambda^{-1} \in \{1, 1/2, \dots\}$ if

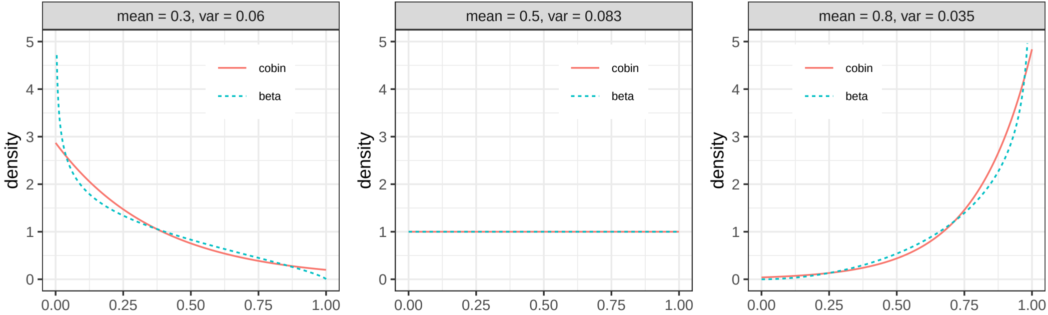
$$p_{\mathrm{cobin}}(y; \theta, \lambda^{-1}) = h(y, \lambda) \exp\left[\lambda\{\theta y - B(\theta)\}\right] = h(y, \lambda) \frac{e^{\lambda\theta y}}{\{(e^{\theta} - 1)/\theta\}^{\lambda}}, \quad 0 \le y \le 1$$

- Exponential dispersion model [Jørgensen, 1987]; $E(Y) = B'(\theta)$, $\mathrm{var}(Y) = \lambda^{-1} B''(\theta)$.
- λ must be an integer to be a valid distribution
- Name motivated from continuous Bernoulli [Loaiza-Ganem and Cunningham, 2019] ($\lambda = 1$ case),

$$Y_1, \dots, Y_\lambda \overset{\text{iid}}{\sim} \mathrm{cobin}(\theta, 1) \implies \tfrac{1}{\lambda} \textstyle\sum_{l=1}^{\lambda} Y_l \sim \mathrm{cobin}(\theta, \lambda^{-1})$$

Cobin vs beta

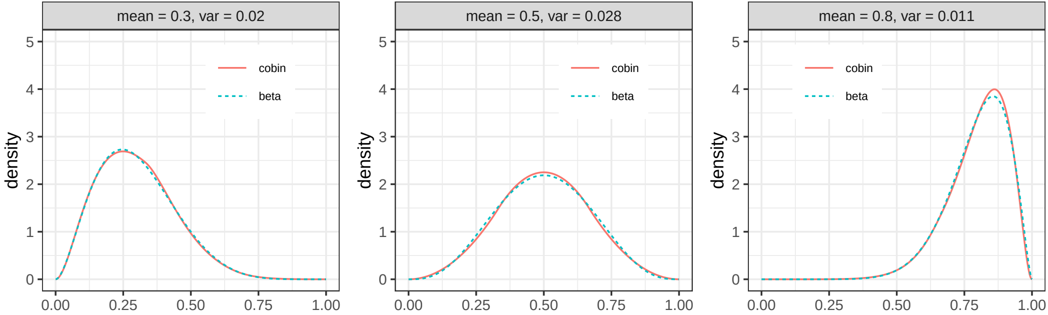
Cobin with $\lambda = 1$; comparison with beta under the same mean and variance



Support: closed interval $[0, 1]$ for cobin with $\lambda = 1$ (continuous Bernoulli)

Cobin vs beta

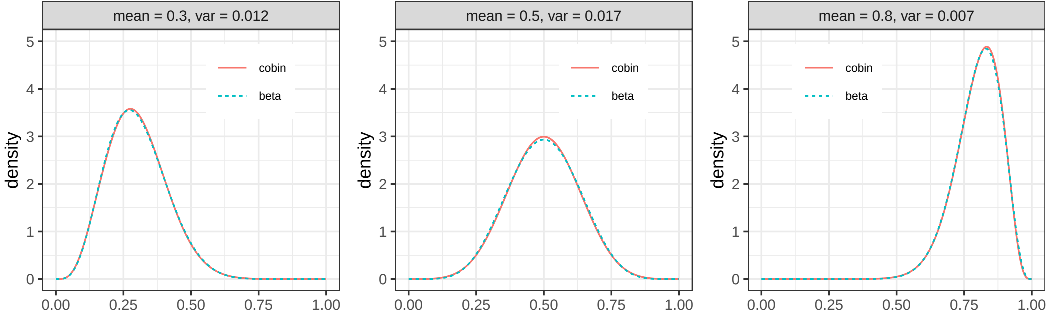
Cobin with $\lambda = 3$; comparison with beta under the same mean and variance



Support: open interval $(0, 1)$ for cobin with $\lambda > 1$

Cobin vs beta

Cobin with $\lambda = 5$; comparison with beta under the same mean and variance



Support: open interval $(0, 1)$ for cobin with $\lambda > 1$

Cobin regression model

- With a choice of link function $g : (0,1) \to \mathbb{R}$,

$$Y_i \mid \theta_i, \lambda \overset{\text{ind}}{\sim} \text{cobin}(\theta_i, \lambda^{-1}), \quad \theta_i = (B')^{-1}\{g^{-1}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta})\}, \quad i = 1, \ldots, n, \tag{1}$$

implies $E(Y_i \mid \boldsymbol{x}_i) = g^{-1}(\eta_i) = g^{-1}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta})$. **Proper GLM** [Nelder and Wedderburn, 1972]

- Likelihood equations (score function):

$$\frac{\partial}{\partial \beta_j}\sum_{i=1}^n \log p_{\text{cobin}}(y_i; \theta_i, \lambda^{-1}) = \lambda \sum_{i=1}^n \frac{(y_i - \mu_i)x_{ij}}{B''(\theta_i)}\frac{\partial \mu_i}{\partial \eta_i} = 0, \quad j = 1, \ldots, p \tag{2}$$

cf. beta score function: $\phi \sum_{i=1}^n [\log \frac{y_i}{1-y_i} - \Psi(\mu_i\phi) + \Psi(\phi - \mu_i\phi)] x_{ij} \frac{\partial \mu_i}{\partial \eta_i}$. ($\Psi$: digamma ft).

Proposition (consistency of cobin MLE under potential misspecification)

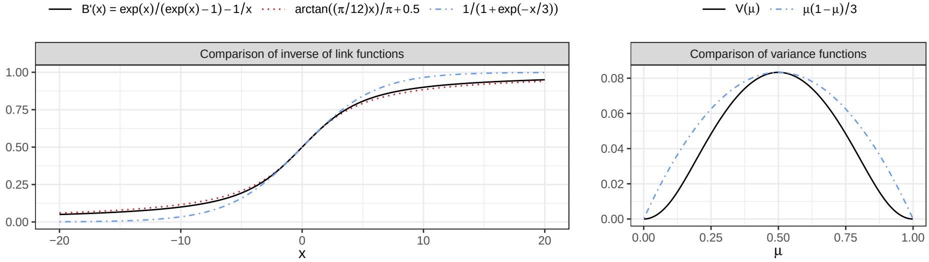
If the mean structure $E(y_i \mid x_i) = g^{-1}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta})$ is correctly specified, the solution $\hat{\beta}$ of the cobin regression likelihood equations (2) is **consistent** [Gourieroux et al., 1984]

Cobin regression with canonical link function

- Canonical link function: $g_{\text{cobit}} = (B')^{-1}$ (“cobit”) that leads to $\theta_i = \eta_i = \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}$
  - Cobin regression likelihood becomes

$$\prod_{i=1}^{n} p_{\text{cobin}}(y_i; \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, \lambda^{-1}) = \prod_{i=1}^{n} h(y_i, \lambda)\frac{e^{\lambda y_i \eta_i}}{\{(e^{\eta_i}-1)/\eta_i\}^{\lambda}}$$

  - $g_{\text{cobit}}^{-1}(\eta) = e^{\eta}/(e^{\eta}-1) - 1/\eta$ is similar to Cauchy c.d.f.



Micobin: dispersion mixtures of cobin distributions

- Limitations of cobin:
  - λ must be an integer to be a valid distribution $\implies$ **limited flexibility**
  - **Unnatural to model dispersion** λ with a separate set of covariates due to discreteness
  - Unless $\lambda = 1$, cobin is supported on open interval $(0, 1)$, **cannot handle exact 0s and 1s**

Definition 2. (micobin)

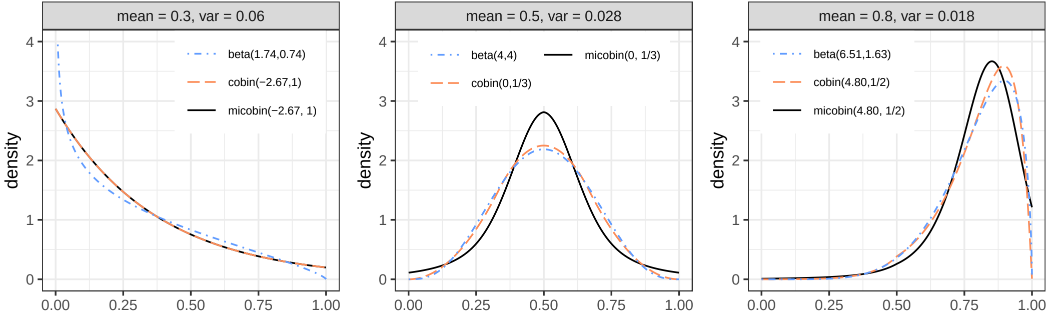
We say $Y \sim \mathrm{micobin}(\theta, \psi)$, natural param. $\theta \in \mathbb{R}$, dispersion $\psi \in (0, 1)$ if

$$Y \mid \lambda \sim \mathrm{cobin}(\theta, \lambda^{-1}), \quad (\lambda - 1) \sim \mathrm{negbin}(2, \psi)$$

- Dispersion $\psi \in (0, 1)$, mean structure preserved $E(Y) = B'(\theta)$ by $\psi-$mixture.
- $(\lambda - 1) \sim \mathrm{negbin}(2, \psi)$ leads to $\mathrm{var}(Y) = \psi B''(\theta)$. (cf. $\mathrm{var}(Y) = \lambda^{-1} B''(\theta)$ for cobin)
- Localizing dispersion ($\lambda \to \lambda_i$), general approach for **robustification** [Wang and Blei, 2018]

Micobin: dispersion mixtures of cobin distributions

Comparison of beta, cobin, and micobin with the same mean and variance.



Support of micobin is a closed interval $[0, 1]$ for any θ, ψ.

Micobin regression and extensions

- Micobin regression with cobit link $E(Y_i \mid \boldsymbol{x}_i) = g_{\text{cobit}}^{-1}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta})$:

$$Y_i \mid \boldsymbol{x}_i \sim \operatorname{micobin}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, \psi) \iff Y_i \mid \boldsymbol{x}_i, \lambda_i \sim \operatorname{cobin}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, \lambda_i^{-1}), \quad (\lambda_i - 1) \sim \operatorname{negbin}(2, \psi)$$

- Can be extended to accommodate dispersion covariate: $\operatorname{logit}(\psi_i) = \boldsymbol{d}_i^{\mathrm{T}}\gamma_i$
- Further hierarchical extensions (with cobit link):
  - **Random intercept model**

$$Y_{ij} \sim \operatorname{micobin}(\boldsymbol{x}_{ij}^{\mathrm{T}}\boldsymbol{\beta} + u_i, \psi), \quad u_i \overset{\text{iid}}{\sim} N(0, \sigma_u^2)$$

  - **Spatial regression model** with spatially indexed data $(y(s_i), \boldsymbol{x}(s_i))$:

$$\begin{aligned} Y(s_i) &\sim \operatorname{micobin}(\eta_i, \psi) \\ \eta_i &= \boldsymbol{x}(s_i)^{\mathrm{T}}\boldsymbol{\beta} + u(s_i), \quad u(\cdot) \sim \text{mean zero GP}. \end{aligned}$$

Data augmentation

- Recall the cobin regression likelihood under the canonical link $\theta_i = \eta_i = \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}$

$$p(y_i \mid \boldsymbol{x}_i, \boldsymbol{\beta}, \lambda) = p_{\text{cobin}}(y_i; \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, \lambda^{-1}) = h(y_i, \lambda)\frac{e^{\lambda y_i \eta_i}}{\{(e^{\eta_i} - 1)/\eta_i\}^{\lambda}}$$

- Not a familiar expression in terms of η_i
- For latent Gaussian models, we desire a log-likelihood that is a quadratic function in η_i
- **Data augmentation**: introduce (missing) auxiliary data $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_n)$ such that
  - Retain original model upon marginalization: $\int p(y_i, \kappa_i \mid \boldsymbol{x}_i, \boldsymbol{\beta}, \lambda)\mathrm{d}\kappa_i = p(y_i \mid \boldsymbol{x}_i, \boldsymbol{\beta}, \lambda)$
  - Conditional distributions $p(y_i \mid \kappa_i, \boldsymbol{x}_i, \boldsymbol{\beta}, \lambda)$, $p(\kappa_i \mid y_i, \boldsymbol{x}_i, \boldsymbol{\beta}, \lambda)$ are easy to work with.

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Kolmogorov-Gamma random variables

Definition 3. (Kolmogorov-Gamma)

We define $\kappa \sim \mathrm{KG}(b, c)$ as an infinite convolution of independent gammas:

$$\kappa \overset{\mathrm{d}}{=} \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{\epsilon_k}{k^2 + c^2/(4\pi^2)}, \quad \epsilon_k \overset{\mathrm{iid}}{\sim} \mathrm{Gamma}(b, 1), \quad k = 1, 2, \ldots$$

- $\pi(\mathrm{KG}(1, 0))^{1/2}$ is same as Kolmogorov distribution [Andrews and Mallows, 1974].
- Comparison with Pólya-Gamma $\omega \sim \mathrm{PG}(b, c)$ [Polson et al., 2013]:

$$\omega \overset{\mathrm{d}}{=} \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{\epsilon_k}{(k - 0.5)^2 + c^2/(4\pi^2)}, \quad \epsilon_k \overset{\mathrm{iid}}{\sim} \mathrm{Gamma}(b, 1), \quad k = 1, 2, \ldots$$

Kolmogorov-Gamma augmentation

Theorem 1. (Kolmogorov-Gamma integral identity)

For any $a \in \mathbb{R}$, $b > 0$, and $\eta \in \mathbb{R}$,

$$\frac{(e^\eta)^a}{\{(e^\eta - 1)/\eta\}^b} = e^{(a-b/2)\eta} \int_0^\infty e^{-\kappa\eta^2/2} p_{\mathsf{KG}}(\kappa; b, 0)\mathrm{d}\kappa, \tag{3}$$

where $p_{\mathsf{KG}}(\kappa; b, 0)$ is the density of a $\mathrm{KG}(b, 0)$ random variable.

- Comparision with Pólya-Gamma integral identity for logistic models [Polson et al., 2013]:

$$\frac{(e^\eta)^a}{(e^\eta + 1)^b} = 2^{-b} e^{(a-b/2)\eta} \int_0^\infty e^{-\omega\eta^2/2} p_{\mathsf{PG}}(\omega; b, 0)\mathrm{d}\omega, \tag{4}$$

- Also, $p_{\mathsf{KG}}(\kappa; b, c) \propto \exp(-c^2\kappa/2) p_{\mathsf{KG}}(\kappa; b, 0)$

Conditional conjugacy with latent Gaussian models

- Consider an augmented model with $\eta_i = \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}$:

$$p_{\text{aug}}(y_i, \kappa_i \mid \eta_i) = h(y_i, \lambda) \exp\left(\lambda(y_i - 0.5)\eta_i - \kappa_i\eta_i^2/2\right) p_{\mathsf{KG}}(\kappa_i; \lambda, 0), \quad i = 1, \ldots, n$$

- By Theorem 1, it recovers cobin regression model upon marginalizing out κ_i. Also,

$$p(\kappa_i \mid \eta_i, y_i) = p_{\mathsf{KG}}(\kappa_i; \lambda, \eta_i)$$
$$p(y_i \mid \kappa_i, \eta_i) \propto N(\lambda(y_i - 0.5)\kappa_i^{-1}; \eta_i, \kappa_i^{-1}) \text{ in terms of } \eta_i$$

- **Offers conditional conjugacy** for normal prior models & latent Gaussian models
  - (Scale mixture of) Normal prior of $\boldsymbol{\beta}$, e.g. $\eta_i = \boldsymbol{x}(s_i)^{\mathrm{T}}\boldsymbol{\beta}$
  - Spatial regression model, e.g. $\eta_i = \boldsymbol{x}(s_i)^{\mathrm{T}}\boldsymbol{\beta} + u(s_i)$, $u(\cdot) \sim \text{Gaussian Process.}$
- Same strategy can be applied to micobin, replacing λ to λ_i

Blocked Gibbs sampler for cobin regression

- $Y_i \sim \text{cobin}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, \lambda^{-1})$, $i = 1, \dots, n$ with normal prior $\boldsymbol{\beta} \sim N(\mathbf{0}, \Sigma_\beta)$

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1. Sample λ from $\text{pr}(\lambda = l \mid \boldsymbol{\beta}) \propto p_\lambda(l) \prod_{i=1}^n p_{\text{cobin}}(y_i; \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, l^{-1})$, $l = 1, \dots, L$
2. Sample κ_i from $(\kappa_i \mid \lambda, \boldsymbol{\beta}) \overset{\text{ind}}{\sim} \text{KG}(\lambda, \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta})$, $i = 1, \dots, n$
3. Sample $\boldsymbol{\beta}$ from $(\boldsymbol{\beta} \mid \lambda, \boldsymbol{\kappa}) \sim N_p(\boldsymbol{m}_\beta, V_\beta)$, where

$$V_\beta^{-1} = X^{\mathrm{T}}\text{diag}(\kappa_1, \dots, \kappa_n)X + \Sigma_\beta^{-1}, \quad \boldsymbol{m}_\beta = V_\beta X^{\mathrm{T}}(y_1\lambda - 0.5\lambda, \dots, y_n\lambda - 0.5\lambda)^{\mathrm{T}}$$

- Some proper prior p_λ for λ and some large upper bound L of λ
- Steps 1,2 jointly updates $(\lambda, \boldsymbol{\kappa})$

Blocked Gibbs sampler for micobin regression

- $Y_i \sim \text{micobin}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, \psi)$, $i = 1, \ldots, n$ with priors $\boldsymbol{\beta} \sim N(\mathbf{0}, \Sigma_\beta)$ and $\psi \sim \text{Beta}(a_\psi, b_\psi)$

---

1. Sample λ_i from $\text{pr}(\lambda_i = l \mid \boldsymbol{\beta}, \psi) \propto l(1-\psi)^{l-1} p_{\text{cobin}}(y_i; \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}, l^{-1})$, $l = 1, \ldots, L$, $i = 1, \ldots, n$
2. Sample κ_i from $(\kappa_i \mid \lambda_i, \boldsymbol{\beta}) \overset{\text{ind}}{\sim} \text{KG}(\lambda_i, \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta})$, $i = 1, \ldots, n$
3. Sample $\boldsymbol{\beta}$ from $(\boldsymbol{\beta} \mid \boldsymbol{\lambda}, \boldsymbol{\kappa}) \sim N_p(\boldsymbol{m}_\beta, V_\beta)$, where
$$V_\beta^{-1} = X^{\mathrm{T}}\text{diag}(\kappa_1, \ldots, \kappa_n)X + \Sigma_\beta^{-1}, \quad \boldsymbol{m}_\beta = V_\beta X^{\mathrm{T}}(y_1\lambda_1 - 0.5\lambda_1, \ldots, y_n\lambda_n - 0.5\lambda_n)^{\mathrm{T}}$$
4. Sample ψ from $(\psi \mid \boldsymbol{\lambda}) \sim \text{Beta}(a_\psi + 2n, b_\psi - n + \sum_{i=1}^n \lambda_i)$

- Steps 1,2 jointly updates $(\lambda, \boldsymbol{\kappa})$, steps 3,4 jointly updates $(\boldsymbol{\beta}, \psi)$

Posterior computation: theory

Theorem 2. (Rapid mixing of Markov chain)

The blocked Gibbs samplers for cobin and micobin regressions are uniformly ergodic. That is, there exist a constant $M > 0$ and $\rho \in [0, 1)$, both independent of initial state, such that $\|P^t(\boldsymbol{\Theta}^{(0)}, \cdot) - \Pi(\cdot)\|_{\mathrm{TV}} \leq M\rho^t$ for all $t \geq 1$.

- Proof by establishing uniform minorization condition, similar to [Choi and Hobert, 2013]
- Guarantees the existence of CLT for Monte Carlo averages of functions of $\boldsymbol{\beta}$
- Strong result for micobin since likelihood is not log-concave

Sampling Kolmogorov-Gamma random variable

$$\kappa \sim \mathrm{KG}(b,c) \iff \kappa \overset{\mathrm{d}}{=} \frac{1}{2\pi^2}\sum_{k=1}^{\infty} \frac{\epsilon_k}{k^2 + c^2/(4\pi^2)}, \quad \epsilon_k \overset{\mathrm{iid}}{\sim} \mathrm{Gamma}(b,1), \quad k = 1, 2, \ldots$$

- Truncating infinite sum of independent gammas: slow, prone to truncation error
- We have $\mathrm{KG}(\lambda, c) \overset{\mathrm{d}}{=} \sum_{l=1}^{\lambda} \mathrm{KG}(1,c)$ for integer λ
- Exact sampling of $\mathrm{KG}(1,c)$: alternating series method [Devroye, 1986]
- Density of $\mathrm{KG}(1,c)$: $p_{\mathsf{KG}}(x;1,c) = \sum_{n=0}^{\infty}(-1)^n a_n(x;c,t)$ with cutoff $t \in (0.0234, 0.25)$,

$$a_n(x;c,t) = \begin{cases} \{\sinh(c/2)/(c/2)\}\exp(-c^2x/2)a_n^L(x), & 0 < x < t, \\ \{\sinh(c/2)/(c/2)\}\exp(-c^2x/2)a_n^R(x), & t \le x \end{cases} \tag{5}$$

- $a_n^L(x)$, $a_n^R(x)$ derived from dual density representation of Kolmogorov r.v. [Feller, 1948]

Sampling Kolmogorov-Gamma $(1, c)$ random variable

1. For $A^L(c,t) = \int_0^t a_0(x;c,t)\mathrm{d}x$ and $A^R(c,t) = \int_t^\infty a_0(x;c,t)\mathrm{d}x$, propose

$$X \sim \begin{cases} \mathrm{GIG}(-1.5, c^2, 1/4)1(0 < X < t) & \text{with prob. } A^L(c,t)/\{A^L(c,t) + A^R(c,t)\} \\ \mathrm{Exp}(c^2/2 + 2\pi^2)1(t \leq X) & \text{with prob. } A^R(c,t)/\{A^L(c,t) + A^R(c,t)\} \end{cases} \tag{6}$$

2. Generate $U \sim \mathrm{Unif}(0, a_0(X;c,t))$
3. Repeat until $U \leq \sum_{n=0}^m (-1)^n a_n(X;c,t)$ (odd m) or $U > \sum_{n=0}^m (-1)^n a_n(X;c,t)$ (even m)
4. Accept X if m is odd, repeat from step 1 again if m is even.

Proposition (KG sampler is fast)

Using the best cutoff point $t^* \approx 0.050239$, the expected number of outer loop & inter loop iterations are bounded above by 1.1456 and 1.1275 for any given c.

Simulation 1: consistency of point estimate

Proposition (consistency of cobin MLE under potential misspecification)

If the mean structure $E(y_i \mid x_i) = g^{-1}(\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta})$ is correctly specified, the solution $\hat{\boldsymbol{\beta}}$ of the cobin regression likelihood equations (2) is **consistent** [Gourieroux et al., 1984]

- Data generation with cobit/logit link functions and 4 different distributions
  - $Y_i \sim \text{beta}(\mu_i, \phi)$, $g(\mu_i) = \beta_0 + \beta_1 x_i$
  - $Y_i \sim \text{cobin}((B')^{-1}(\mu_i), \lambda^{-1})$, $g(\mu_i) = \beta_0 + \beta_1 x_i$
  - $Y_i \sim \text{beta rectangular}(\mu_i, \alpha, \phi) = w_i \text{beta}(\tilde{\mu}_i, \phi) + (1 - w_i)\text{unif}(0, 1)$, $g(\mu_i) = \beta_0 + \beta_1 x_i$
  - $Y_i \sim 0.25\text{beta}(\mu_i - \epsilon_i, \phi) + 0.5\text{beta}(\mu_i, \phi) + 0.25\text{beta}(\mu_i + \epsilon_i, \phi)$, $g(\mu_i) = \beta_0 + \beta_1 x_i$
- Correct link & mean structure $g(E(Y_i \mid x_i)) = \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{\beta}$, but distribution can be misspecified
- $n \in \{100, 400, 1600\}$, $(\beta_0^{\text{true}}, \beta_1^{\text{true}}) = (0, 1)$, compare betareg $\hat{\boldsymbol{\beta}}$ and cobinreg $\hat{\boldsymbol{\beta}}$

Simulation 1 results

			Beta		Cobin		Beta rectangular		Mixture of beta	
Link	Method	n	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
cobit	beta regression	100	**0.004**	**0.106**	-0.024	0.099	-0.061	0.153	-0.030	0.094
		400	-0.003	**0.054**	-0.030	0.057	-0.069	0.099	-0.037	0.058
		1600	0.002	**0.026**	-0.030	0.038	-0.076	0.084	-0.036	0.042
	cobin regression	100	0.005	0.113	**0.003**	0.099	**0.013**	**0.134**	**0.006**	**0.092**
		400	**-0.002**	0.056	**-0.001**	**0.049**	**0.006**	**0.070**	**-0.001**	**0.046**
		1600	0.002	0.027	**0.000**	**0.023**	**-0.001**	**0.035**	**0.001**	**0.022**
logit	beta regression	100	**0.003**	**0.084**	-0.043	0.080	-0.054	0.117	-0.041	0.074
		400	**0.000**	**0.042**	-0.047	0.059	-0.059	0.080	-0.046	0.055
		1600	0.000	**0.021**	-0.045	0.048	-0.062	0.068	-0.046	0.048
	cobin regression	100	0.015	0.101	**0.005**	**0.066**	**0.020**	**0.116**	**0.005**	**0.067**
		400	0.004	0.051	**0.000**	**0.035**	**0.007**	**0.062**	**0.000**	**0.033**
		1600	0.000	0.026	**0.001**	**0.016**	**0.001**	**0.032**	**0.001**	**0.016**

- Cobin regression $\hat{\beta}$ is consistent even under the misspecified distribution

Simulation 2: spatial regression

- Resembling spatially indexed MMI data $Y(s_i)$
- Data generation: $Y(s_i) \sim \text{beta rectangular}(\mu_i, \alpha, \phi)$, locations $s_i \in [0,1]^2$ uniformly, and

$$g_{\text{cobit}}(\mu_i) = \beta_0 + \beta_1 x(s_i) + u(s_i), \quad u(\cdot) \sim \text{mean zero Gaussian process}$$

- $(\beta_0^{\text{true}}, \beta_1^{\text{true}}) = (0, 1)$, $\rho \in \{0.1, 0.2\}$ (spatial dependence)
- Fit data with (1) spatial beta, (2) spatial cobin, (3) spatial micobin regression models
  - **All models are misspecified**
- `Stan` for spatial beta; Gibbs sampler for spatial cobin/micobin; 5000 MCMC samples.
- Compare (1) Inference of β, (2) predictive performance, (3) sampling performance

Simulation 2 results

ρ	Method	$(n_{\text{train}}, n_{\text{test}})$	Inference ($\hat{\beta}_1$)		Prediction		Sampling ($\boldsymbol{\beta}$)	
			Bias	RMSE	negtestLL	MSPE$\times 10^2$	mESS	time (min)
0.1	beta regression	(200, 50)	-0.048	0.118	-0.325	0.427	919.8	44.5
		(400, 100)	-0.052	0.089	-0.354	0.345	978.7	437.7
	cobin regression	(200, 50)	**0.005**	**0.093**	-0.340	0.388	2791.3	2.0
		(400, 100)	**0.005**	**0.067**	-0.372	0.323	3220.9	11.2
	micobin regression	(200, 50)	0.034	0.099	**-0.367**	**0.373**	1908.4	2.4
		(400, 100)	0.037	0.074	**-0.394**	**0.312**	2137.5	11.7
0.2	beta regression	(200, 50)	-0.065	0.120	-0.320	0.329	1187.2	96.3
		(400, 100)	-0.052	0.095	-0.350	0.248	808.0	933.4
	cobin regression	(200, 50)	**0.000**	**0.088**	-0.346	0.306	3366.0	2.2
		(400, 100)	**0.013**	**0.078**	-0.370	0.233	3663.9	12.1
	micobin regression	(200, 50)	0.039	0.092	**-0.373**	**0.293**	2265.3	2.2
		(400, 100)	0.050	0.091	**-0.395**	**0.226**	2575.4	12.7

Monte Carlo standard errors are all less than 0.015 for negtestLL, 0.013 for MSPE, 127.2 for mESS.

- Cobin gives lowest bias/RMSE of $\hat{\beta}_1$, micobin achieves best predictive performance
- Multivariate ESS per time: cobin and micobin better than 40x or more

Benthic macroinvertebrate multimetric index (MMI)

- Index between 0 and 100, scaled by 0.01
- a.k.a. index of biotic integrity [Karr, 1991]
- Combines various macroinvertebrate assemblages attributes
  - Taxonomic composition, richness, presence of sensitive species (e.g. mayfly)...
- Higher MMI indicates a healthier and diverse benthic macroinvertebrate community
- Data from 2017 NLA survey, 950 lakes
- One lake had a 0 MMI value, set $n = 949$ for comparison with beta regression

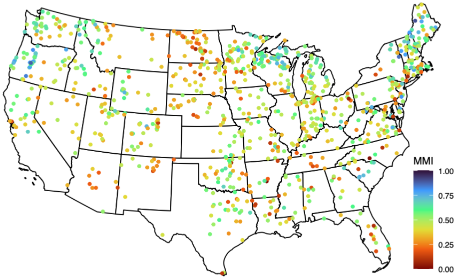


Figure: MMI of 949 US lakes

Lake watershed covariate

- Data source: LakeCat [Hill et al., 2018]
- Various lake catchment/watershed information of 300k+ US lakes
  - For prediction only, we use 55215 lakes with surface area $> 40,000m^2$
- 9 watershed covariates, log transformed:
  - agkffact (soil erodibility), bfi (base flow index), conif (coniferous forest cover)
  - cbnf (cultivated N fixation), crophay (crop&hay land cover), fert (synthetic N fertilizer use), manure (manure application)
  - urbmdhi (medium/high density urban land cover), pestic97 (1997 pesticide use)

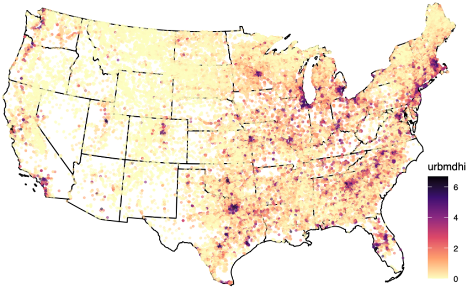


Figure: medium/high density urban land cover percentage, log transformed

Model fit

- Fit three different spatial regression models (beta, cobin, micobin) with

$$g_{\mathrm{cobit}}(\mu_i) = \boldsymbol{x}(s_i)^{\mathrm{T}}\boldsymbol{\beta} + u(s_i), \quad u(\cdot) \sim \text{nearest neighbor GP}$$

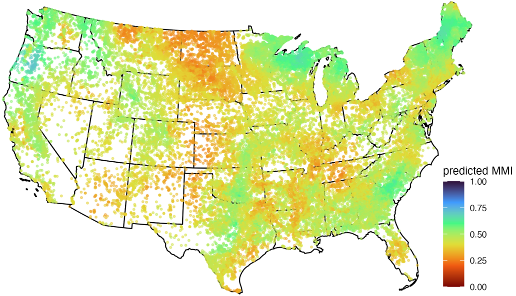
- Employ NNGP [Datta et al., 2016] due to prediction at 50k+ locations
- Prior $\boldsymbol{\beta} \sim N_p(\mathbf{0}, 100^2 I_p)$, half-Cauchy on random effect standard deviation
- `Stan` for spatial beta; Gibbs for spatial cobin/micobin; 5000 MCMC samples, 3 chains
  - Leveraging normal conjugacy via KG augmentation, we jointly update $\boldsymbol{\beta}$ and $\{u(s_i)\}_{i=1}^n$ by partial collapsing [Van Dyk and Park, 2008]
  - Took 2 hr for spatial beta, **5 min for cobin and micobin** per chain
  - mESS/time difference more than 20x.

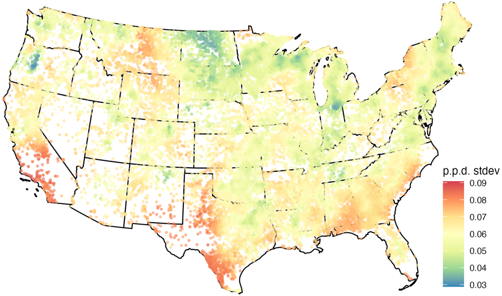
MMI data analysis results: association

	Beta regression		Cobin regression		Micobin regression	
Variable	Estimate	95% CI	Estimate	95% CI	Estimate	95% CI
Intercept	-2.363	(-4.160, -0.553)	-2.106	(-3.859, -0.345)	-1.797	(-3.551, -0.085)
agkffact	-2.586	(-5.584, 0.330)	**-2.888**	**(-5.714, -0.003)**	**-3.457**	**(-6.113, -0.800)**
bfi	**0.343**	**(0.016, 0.672)**	0.293	(-0.022, 0.614)	0.229	(-0.082, 0.548)
cbnf	0.165	(-0.081, 0.412)	0.182	(-0.055, 0.420)	0.191	(-0.035, 0.425)
conif	0.081	(-0.002, 0.164)	**0.093**	**(0.011, 0.176)**	**0.123**	**(0.044, 0.203)**
crophay	-0.079	(-0.250, 0.091)	-0.063	(-0.231, 0.106)	-0.054	(-0.213, 0.105)
fert	-0.073	(-0.310, 0.158)	-0.092	(-0.323, 0.132)	-0.082	(-0.300, 0.138)
manure	-0.048	(-0.202, 0.102)	-0.036	(-0.182, 0.115)	-0.029	(-0.173, 0.118)
pestic97	-0.014	(-0.106, 0.075)	-0.021	(-0.108, 0.067)	-0.025	(-0.108, 0.059)
urbmdhi	**-0.181**	**(-0.288, -0.076)**	**-0.170**	**(-0.273, -0.067)**	**-0.142**	**(-0.243, -0.041)**

- WAIC: -1093.4 (beta), -1103.5 (cobin), **-1119.3 (micobin)**
- Selected variables based on 95% CI are different for beta

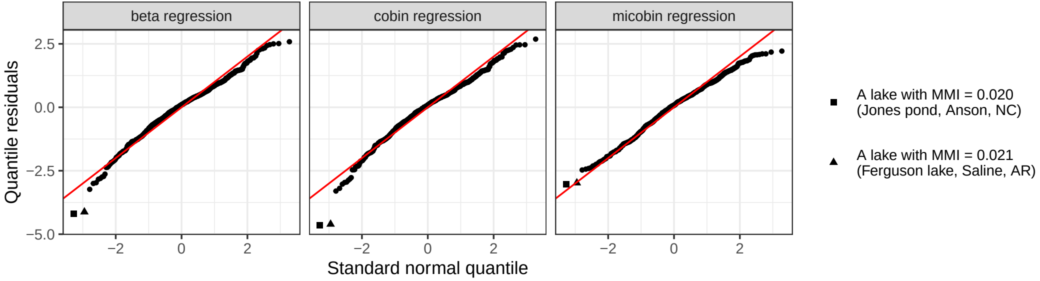
MMI data analysis results: prediction





- Left: predicted MMI, Right: stdev of predicted MMI

MMI data analysis results: goodness of fit



- Quantile residual plot [Dunn and Smyth, 1996]: $\Phi^{-1}(F(y_i \mid \hat{\mu}_i, \hat{\phi}))$ against normal quantiles
- Two influential observations with the lowest MMI values of 0.02 and 0.021
- Re-run the analysis, removing those two lakes

MMI data analysis results: sensitivity analysis

$(n = 947)$	Beta regression		Cobin regression		Micobin regression	
Variable	Estimate	95% CI	Estimate	95% CI	Estimate	95% CI
(Intercept)	-1.829	(-3.621, -0.070)	-1.756	(-3.531, 0.006)	-1.741	(-3.520, 0.018)
agkffact	**-3.088**	**(-5.982, -0.206)**	**-3.150**	**(-6.019, -0.258)**	**-3.494**	**(-6.220, -0.822)**
bfi	0.244	(-0.075, 0.568)	0.228	(-0.088, 0.552)	0.219	(-0.097, 0.540)
cbnf	0.191	(-0.044, 0.430)	0.200	(-0.035, 0.437)	0.196	(–0.034, 0.424)
conif	**0.096**	**(0.014, 0.175)**	**0.103**	**(0.021, 0.183)**	**0.125**	**(0.045, 0.204)**
crophay	-0.057	(-0.223, 0.110)	-0.053	(-0.218, 0.114)	-0.050	(-0.210, 0.110)
fert	-0.096	(-0.327, 0.135)	-0.104	(-0.329, 0.122)	-0.089	(-0.316, 0.135)
manure	-0.001	(-0.148, 0.148)	-0.009	(-0.157, 0.138)	-0.022	(-0.167, 0.122)
pestic97	-0.031	(-0.118, 0.057)	-0.030	(-0.119, 0.057)	-0.027	(-0.110, 0.055)
urbmdhi	**-0.180**	**(-0.283, -0.076)**	**-0.169**	**(-0.275, -0.064)**	**-0.143**	**(-0.242, -0.043)**
Change	$\Vert\hat{\boldsymbol{\beta}}^{(949)} - \hat{\boldsymbol{\beta}}^{(947)}\Vert_2 = 0.743$		$\Vert\hat{\boldsymbol{\beta}}^{(949)} - \hat{\boldsymbol{\beta}}^{(947)}\Vert_2 = 0.444$		$\Vert\hat{\boldsymbol{\beta}}^{(949)} - \hat{\boldsymbol{\beta}}^{(947)}\Vert_2 = 0.069$	

- Beta regression results changed, cobin/micobin results remained same
- Recall that beta score function is unbounded in y

MMI data analysis results: sensitivity analysis

	Micobin regression ($n = 950$)		Micobin regression ($n = 949$)	
Variable	Estimate	95% CI	Estimate	95% CI
(Intercept)	-1.758	(-3.517, -0.04)	-1.797	(-3.551, -0.085)
agkffact	**-3.456**	**(-6.175, -0.794)**	**-3.457**	**(-6.113, -0.800)**
bfi	0.219	(-0.100, 0.537)	0.229	(-0.082, 0.548)
cbnf	0.187	(-0.040, 0.415)	0.191	(-0.035, 0.425)
conif	**0.128**	**(0.048, 0.208)**	**0.123**	**(0.044, 0.203)**
crophay	-0.060	(-0.222, 0.101)	-0.054	(-0.213, 0.105)
fert	-0.071	(-0.296, 0.13)	-0.082	(-0.300, 0.138)
manure	-0.031	(-0.178, 0.115)	-0.029	(-0.173, 0.118)
pestic97	-0.023	(-0.106, 0.059)	-0.025	(-0.108, 0.059)
urbmdhi	**-0.141**	**(-0.243, -0.038)**	**-0.142**	**(-0.243, -0.041)**

- Micobin does not require removing a datum with MMI = 0
- Result almost unchanged

- Preprint: `https://arxiv.org/abs/2504.15269`
- Reproducing code: `https://github.com/changwoo-lee/cobin-reproduce`
- R package ”cobin”: `https://github.com/changwoo-lee/cobin`

Thank you!

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